We note that the formulas given above, describing unloading shock waves, are valid only if $\lambda_1^0 < \lambda_m$ (see Fig. 2a'). In the contrary case, where $\lambda_1^0 > \lambda_m$, at the start of the unloading process, shock waves will not exist, and only in a certain time after the start of the unloading process, in a medium with a sufficiently small relaxation time (or for a very long rod), will the weak wave arising at the start go over into a shock wave, whose intensity will than fall further with time.

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DISPERSION OF THE VELOCITY AND SCATTERING OF ULTRASONIC WAVES IN COMPOSITE MATERIALS

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UDC 534.514

The scattering of waves at the inhomogeneities of a medium can be calculated by various methods. An analysis of the most frequently used approximations was made in [1, 2]. The scattering coefficient of an ultrasonic wave in compositite materials was calculated in [3-6]. In [3], the smallness of one of the components was assumed, while, in [6], only the asymptote of long and short waves were calculated. An attempt at the calculation of the scattering coefficient of longitudinal and transverse ultrasonic waves over the whole range of wavelengths was made in [4, 5]. The calculation was made under the approximation of taking account of pairwise correlations between the moduli of elasticity and the density. In [4], the calculations were made using a Gaussian distribution for the coordinate parts of the binary correlation functions, which does not relate to composite materials, and, in [5], the explicit form of a function enabling a transition from asymptote of long waves to a short-wave asymptote is not given. In addition, neither of the above-cited pieces of work took into consideration the distribution of the velocity of the propagating wave.

A calculation of the scattering coefficient and the dispersion of the velocity of longitudinal waves over the whole range of wavelengths, with arbitrary concentrations of the components, is given below.

§ 1. We renormalize the equations of motion using a method developed in [7-9]:

$$L_{il}u_l = 0,\ L_{il} \equiv \nabla_k \lambda_{iklm} \nabla_m + \rho\omega^2\delta_{il},$$

where u is the vector of the displacement; λ_{iklm} is the tensor of the moduli of elasticity; ρ is the density of the medium; ω is the cyclic frequency.

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 145-152, May-June, 1978. Original article submitted May 3, 1977.

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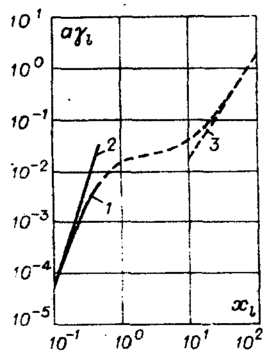

Fig. 1

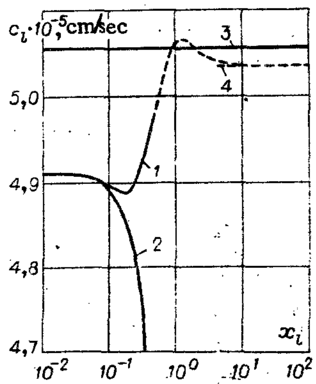

Fig. 2

Denoting the regular components of the operators and functions by angular brackets, and the random components by primes, in the second approximation of the theory of random functions we find

$$\langle L_{il}\rangle\langle u_l\rangle + \langle L'_{ip}(\mathbf{r})\int G_{ps}(\mathbf{r}-\mathbf{r}_1)L'_{sl}(\mathbf{r}_1)\rangle\langle u_l(\mathbf{r}_1)\rangle\, d\,\mathbf{r}_1 = 0, \tag{1.1}$$

where $L'_{ip} = L_{ip} - \langle L_{ip}\rangle$; G_{ps} is a Green function of the regular operator $\langle L_{il}\rangle$, determined by the equality

$$\langle L_{ip}\rangle G_{ps} = -\,\delta(\mathbf{r})\delta_{is}.$$

Here and in what follows, the averaging is carried out over a region far exceeding the spatial scale of the correlations, but considerably less than the distances at which the regular components of the functions change appreciably.

From Eq. (1.1) it can be seen that the regular component of the vector of the displacement for a plane wave can be written in the form

$$\langle u_l(\mathbf{r})\rangle = A_l(\mathbf{q})e^{-i\mathbf{q}\mathbf{r}}, \tag{1.2}$$

where $A_l(\mathbf{q})$ is the amplitude of the wave for the mean vector of the displacement; $\mathbf{q}$ is the wave vector; and the factor $e^{i\omega t}$ is omitted due to the homogeneity of Eq. (1.1).

Taking the value of $\langle u_l\rangle$ out from under the integral sign using relationship (1.2), we find the renormalized operator of the equation of motion L^*_{il}:

$$L^*_{il}\langle u_l\rangle = 0,\; L^*_{il} = \langle L_{il}\rangle + \tilde{L}_{il},\; \langle L_{il}\rangle = \langle\lambda_{iklm}\rangle\nabla_{km} + \omega^2\langle\rho\rangle\delta_{il};$$
$$\tilde{L}_{il} = A^{rslm}_{ikpq}I_{pqrs}\nabla_{km} - 2i\,\omega^2 A_{rsm(l}I_{i)rs}\nabla_m + \omega^4 A I_{il}. \tag{1.3}$$

Here $\nabla_{km} \equiv \nabla_k\nabla_m$, and symmetrization is carried out with respect to the indices in parentheses. The operator (1.3) was written under the assumption of the separation of the tensor and coordinate dependences of the binary correlation functions; here, for different correlation functions, an identical correlation dependence is taken:

$$\langle\lambda'_{iklm}(\mathbf{r}_1)\lambda'_{pqrs}(\mathbf{r}_1+\mathbf{r})\rangle = A^{iklm}_{pqrs}\,\varphi(\mathbf{r}),$$
$$\langle\lambda'_{iklm}(\mathbf{r}_1)\rho'(\mathbf{r}_1+\mathbf{r})\rangle = A_{iklm}\varphi(\mathbf{r}),\; \langle\rho'(\mathbf{r}_1)\rho'(\mathbf{r}_1+\mathbf{r})\rangle = A\,\varphi(\mathbf{r}).$$

The integrals I are determined by the following dependences:

$$I_{pqrs} = \int G_{pr}\nabla_{qs}(\varphi\cos\mathbf{q}\mathbf{r})\,d\,\mathbf{r},$$
$$I_{pql} = \int G_{pl}\nabla_q(\varphi\sin\mathbf{q}\mathbf{r})\,d\mathbf{r},\quad I_{il} = \int G_{il}\varphi\cos\mathbf{q}\mathbf{r}\,d\mathbf{r}.$$

The correlation tensors A^{iklm}_{pqrs}, A_{iklm}, and A for composite materials are equal to

$$A^{iklm}_{pqrs} = h_1\delta_{ik}\delta_{lm}\delta_{pq}\delta_{rs} + 2h_2(\delta_{ik}\delta_{lm}\delta_{p(r}\delta_{s)q} + \delta_{i(l}\delta_{m)k}\delta_{pq}\delta_{rs}) + 4h_3\delta_{i(l}\delta_{m)k}\delta_{p(r}\delta_{s)q}, \tag{1.4}$$

$$A_{iklm} = h_4\delta_{ik}\delta_{lm} + 2h_5\delta_{i(l}\delta_{m)k},\; A = h_6;$$
$$h_1 = \langle\lambda'\lambda'\rangle,\; h_2 = \langle\lambda'\mu'\rangle,\; h_3 = \langle\mu'\mu'\rangle, \tag{1.5}$$
$$h_4 = \langle\lambda'\rho'\rangle,\; h_5 = \langle\mu'\rho'\rangle,\; h_6 = \langle\rho'\rho'\rangle,$$

where, for two-phase composite materials, the mean value, the dispersion, and the covariance of arbitrary values of ξ and η are expressed in terms of the volumetric concentrations of the components $\bar{v}_i$ using the relationships

$$\langle\xi\rangle = \bar{v}_1\xi_1 + \bar{v}_2\xi_2,\ \langle\xi'\xi'\rangle = \bar{v}_1\bar{v}_2(\xi_1 - \xi_2)^2,$$
$$\langle\xi'\eta'\rangle = \bar{v}_1\bar{v}_2(\xi_1 - \xi_2)(\eta_1 - \eta_2). \tag{1.6}$$

We assume that the boundaries between the components of the composite material are sharp. Then, if there is no short- or long-range order in the spatial distribution of the components (a completely unordered structure), then the explicit form of the function $\varphi(\mathbf{r})$ can be taken as exponential [7-11]:

$$\varphi(\mathbf{r}) = \varphi(r) = \exp(-r/a), \tag{1.7}$$

where a is the radius of the correlations.

A contribution to the scattering and the dispersion of the velocity is made by the correlation term $\tilde{L}_{il}(\mathbf{r})$. In Fourier space, the operator $\tilde{L}_{il}(\mathbf{r})$ can be represented in the form

$$\tilde{L}_{il}(\mathbf{q}) = -q^2[\Lambda(x)l_{il} + M(x)\delta_{il}],\ l_{il} = l_i l_l,\ l_i = q_i/q. \tag{1.8}$$

Then, analogously to [7], the scattering coefficient and the velocity of the longitudinal waves will be equal to

$$a\gamma_l(x) = \frac{xQ(x)}{2\langle\rho\rangle c^2},\ Q(x) = \mathrm{Im}(\Lambda + M),\ c^2 = c_l^2 = \frac{\langle\lambda + 2\mu\rangle}{\langle\rho\rangle}; \tag{1.9}$$

$$c_l(x) = c\left(1 + \frac{R(x)}{2\langle\rho\rangle c^2}\right),\ R(x) = 1 + x\frac{d}{dx}\mathrm{Re}(\Lambda + M),\ x = qa = \frac{\omega a}{c}, \tag{1.10}$$

where $\Lambda = \Lambda(x)$; $M = M(x)$.

From the latter expressions it follows that to calculate the scattering coefficient of longitudinal ultrasonic waves, it is sufficient to find the function Q(x), and, to find the dispersion of the velocity, the function R(x). To bring the operator $\tilde{L}_{il}$ to the form (1.8), we use expression (1.2). This permits us to transform the operator $\tilde{L}_{il}$:

$$\tilde{L}_{il}(\mathbf{q}) = -q^2(l_{km}A_{ikpq}^{rslm}I_{pqrs} + 2qc^2 l_k A_{ikpq}I_{pql} - q^2c^4 A I_{il}). \tag{1.11}$$

In the second term of the latter expression, symmetrization is not carried out, since the tensor $l_k A_{ikpq} I_{pql}$, in accordance with (1.8), is symmetrical with respect to permutation of the indices i and l.

§ 2. Let us calculate the integrals I_{pqrs}, I_{pql}, and I_{il}. For this purpose, we use a known expression of the Green function G_{pr} [7] and the explicit function $\varphi(\mathbf{r})$, in accordance with (1.7). We also carry out the replacement of variables $\mathbf{q} = q\mathbf{l}$, $\xi = q\mathbf{r}$, and $\mathbf{r} = r\mathbf{n}$. We will then have

$$I_{pqrs} = \int [n_{pr}g(\xi c) + \delta_{pr}f(\xi c)]\frac{\partial^2}{\partial\xi_q\partial\xi_s}\left[\exp\left(-\frac{\xi}{x}\right)\cos(\xi\mathbf{l})\right]\xi d\xi d\Omega_n,$$
$$I_{pql} = \frac{1}{q}\int\left[n_{pl}g(\xi c) + \delta_{pl}f(\xi c)\right]\frac{\partial}{\partial\xi_q}\left[\exp\left(-\frac{\xi}{x}\right)\sin(\xi\mathbf{l})\right]\xi d\xi d\Omega_n, \tag{2.1}$$
$$I_{il} = \frac{1}{q^2}\int [n_{il}g(\xi c) + \delta_{il}f(\xi c)]\exp\left(-\frac{\xi}{x}\right)\cos(\xi\mathbf{l})\,\xi d\xi d\Omega_n;$$

$$g(\xi c) = \frac{1}{4\pi\langle\rho\rangle c^2\xi^2}\{[3(1 + ik\xi) - k^2\xi^2]e^{-ik\xi} - [3(1 + i\xi) - \xi^2]e^{-i\xi}\},$$
$$f(\xi c) = \frac{1}{4\pi\langle\rho\rangle c^2\xi^2}\{(1 + i\xi)e^{-i\xi} - (1 + ik\xi - k^2\xi^2)e^{-ik\xi}\}, \tag{2.2}$$

where

$$n_{pl} = n_p n_l;\ k = c/c_t;\ c_t^2 = \langle\mu\rangle/\langle\rho\rangle;\ d\Omega_n = d\mathbf{r}/r^2dr = d\xi/\xi^2 d\xi. \tag{2.3}$$

The method of calculating the integrals (2.1) is described in [9]. Carrying out the corresponding calculation, we obtain

$$I_{pqrs} = d_1 l_{pqrs} + d_2(l_{pr}\delta_{sq} + l_{ps}\delta_{rq} + l_{pq}\delta_{rs} + l_{rs}\delta_{pq} + l_{rq}\delta_{ps} +$$
$$+ l_{sq}\delta_{pr}) + d_3(\delta_{pq}\delta_{rs} + \delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr}) + d_4 l_{qs}\delta_{pr} + d_5\delta_{pr}\delta_{sq}, \tag{2.4}$$

$$I_{pql} = \frac{1}{2qc^2}[d_6 l_{pql} + d_7(l_p\delta_{lq} + l_q\delta_{pl} + l_l\delta_{pq}) + d_8 l_q\delta_{pl}],\ I_{il} = -\frac{1}{q^2c^4}(d_9 l_{il} + d_{10}\delta_{il}),$$

where $d_n = a_n + ib_n$: the values of a_n and b_n are equal to

$$\langle\rho\rangle c^2 a_1 = \frac{5}{16x^4}\left[-14u + (-23 + 16k^2 + 7k^4)x^2\right] - v_{30}(1 + 5x^2) + \tag{2.5}$$

$$+ v_1 - v_2 v_3 + v_4 v_5, \quad \langle\rho\rangle c^2 a_2 = \frac{1}{16x^4}\left[10u + (13 - 8k^2 - 5k^4)x^2\right] + v_2 v_6 - v_4 v_7,$$

$$\langle\rho\rangle c^2 a_3 = \frac{1}{16x^4}\left[-2u + (k^4 - 1)x^2\right] - v_2 v_8 + v_4 v_9, \quad \langle\rho\rangle c^2 a_4 =$$

$$= -\frac{3k^2}{2x^2} - v_1 + v_2 v_{10}, \quad \langle\rho\rangle c^2 a_5 = \frac{k^2}{2x^2} - v_2 v_{11},$$

$$\frac{\langle\rho\rangle}{2} a_6 = \frac{15u}{8x^2} + v_{30}(1 + 3x^2) - v_{18} + v_2 v_{19} - v_4 v_{20}, \quad \frac{\langle\rho\rangle}{2} a_7 = -\frac{3u}{8x^2} - v_2 v_{21} + v_4 v_{22},$$

$$\frac{\langle\rho\rangle}{2} a_8 = v_{18} - \frac{k^2 v_2}{x}, \quad \frac{\langle\rho\rangle}{c^2} a_9 = v_{23} - v_{30} x^2 - v_2 v_{24} + v_4 v_{25},$$

$$\frac{\langle\rho\rangle}{c^2} a_{10} = -v_{23} + v_2 v_{26} - v_4 v_{27};$$

$$\langle\rho\rangle c^2 b_1 = \frac{5}{48x}(21v_0 + 76 - 21k - 34k^3 - 21k^5) - v_{12} + 2x^3 v_{30} - \tag{2.6}$$

$$- v_3 v_{13} - v_5 v_{14}, \quad \langle\rho\rangle c^2 b_2 = \frac{1}{48x}(-15v_0 - 44 + 15k + 14k^3 + 15k^5) + v_6 v_{13} +$$

$$+ v_7 v_{14}, \quad \langle\rho\rangle c^2 b_3 = \frac{1}{48x}(3v_0 + 4 - 3k + 2k^3 - 3k^5) - v_8 v_{13} - v_9 v_{14}, \quad \langle\rho\rangle c^2 b_4 =$$

$$= \frac{3k^3}{2x} + v_{12} + v_{10} v_{13}, \quad \langle\rho\rangle c^2 b_5 = -\frac{k^3}{2x} - v_{11} v_{13},$$

$$\frac{\langle\rho\rangle}{2} b_6 = -5v_{31} + v_{28} - v_{30} x(1 + 2x^2) + v_{13} v_{19} + v_{14} v_{20},$$

$$\frac{\langle\rho\rangle}{2} b_7 = v_{31} - v_{13} v_{21} - v_{14} v_{22}, \quad \frac{\langle\rho\rangle}{2} b_8 = -v_{28} - \frac{k^2 v_{13}}{x},$$

$$\frac{\langle\rho\rangle}{c^2} b_9 = \frac{3}{2x}(1 - k) + 2x^3 v_{30} - v_{29} - v_{13} v_{24} - v_{14} v_{25},$$

$$\frac{\langle\rho\rangle}{c^2} b_{10} = \frac{1}{2x}(k - 1) + v_{29} + v_{13} v_{26} + v_{14} v_{27},$$

where

$$\begin{gathered}
v = 1 + k^2; \; u = 1 - k^2; \; w = (1 + 2vx^2 + u^2x^4)^{-1}; \\
x^4 v_0 = 1 - k + 2x^2(2 - k - k^3); \; v_1 = wk^2\left[1 + (2 + 3k^2)x^2 + ux^4\right]; \\
2v_2 = \operatorname{arctg}\frac{2x}{1 - ux^2}; \\
16x^7 v_3 = 5[7 + 21vx^2 + 3(7 + 10k^2 + 7k^4)x^4 + \\
+ (7 + 9k^2 + 9k^4 + 7k^6)x^6]; \\
2v_4 = \operatorname{arctg} 2x; \; 16x^7 v_5 = 5(7 + 42x^2 + 72x^4 + 32x^6); \\
16x^7 v_6 = 5 + 15vx^2 + 3(5 + 6k^2 + 5k^4)x^4 + (5 + 3k^2 + \\
+ 3k^4 + 5k^6)x^6; \; 16x^7 v_7 = 5 + 30x^2 + 48x^4 + 16x^6; \\
16x^7 v_8 = 1 + 3vx^2 + (3 + 2k^2 + 3k^4)x^4 + u(1 - k^4)x^6; \\
16x^7 v_9 = 1 + 6x^2 + 8x^4; \; 2x^3 v_{10} = 3k^2(1 + vx^2); \\
3v_{11} = v_{10}; \; v_{12} = 2k^5x^3w; \; 4v_{13} = \ln\{w[1 + (k - 1)^2x^2]^2\}; \\
4v_{14} = -\ln v_{30}; \; v_{18} = wk^2[1 + (1 + v)x^2 + ux^4]; \\
8x^5 v_{19} = 3[5 + 10vx^2 + (5 + 6k^2 + 5k^4)x^4]; \; 8x^5 v_{20} = \\
= 3(5 + 20x^2 + 16x^4); \; 8x^5 v_{21} = 3 + 6vx^2 + (3 + 2k^2 + \\
+ 3k^4)x^4; \; 8x^5 v_{22} = 3 + 12x^2 + 8x^4; \; v_{23} = wk^2x^2(1 + ux^2); \\
k^2 v_{24} = v_{10}; \; 2x^3 v_{25} = 3(1 + 2x^2); \; 3k^2 v_{26} = v_{10}; \; 3v_{27} = v_{25}; \\
v_{28} = wk^3x(1 + vx^2); \; v_{29} = 2k^3x^3w; \; v_{30} = (1 + 4x^2)^{-1}; \\
8x^3 v_{31} = 3[1 - k + (2 - k - k^3)x^2].
\end{gathered} \tag{2.7}$$

In formulas (2.7) of the present article, errors in [9] for P_4 and u_2 are corrected. The correct values for them are obtained from the values of v_4 and v_2 of the present article by the replacements $v_4 = P_4$, $v_2 = u_2$, $x = 1/s$, and $k = \varkappa$; in [9], in formula (2.7), $\rho^2 c b_4$ should read $\rho c^2 b_4$; in formula (2.3), in the expression for R_5 in the last square brackets, $+ J_s - J_s\beta(3 + 2\beta^2)$ should read $+ J_c - J_s\beta(3 + 2\beta^2)$.

§3. We pass on directly to calculation of the functions $Q(x)$ and $R(x)$. For this purpose, we substitute into expression (1.11) the values found for the integrals I_{pqrs}, I_{pql}, and I_{il} (2.4) and make use of expressions (1.4). This makes it possible to represent the operator $\tilde{L}_{il}$ (q) in the form (1.8). Under these circumstances, the functions $Q(x)$ and $R(x)$ are found to be equal to

$$Q(x)=\sum_1^{10} D_n b_n,\quad R(x)=\sum_1^{10} D_n \chi_n, \tag{3.1}$$

where

$$\chi_n \equiv (1+xd/dx)a_n;\ D_1=h_1+4h_2+4h_3;$$
$$D_2=2(5h_1+16h_2+12h_3);\ D_3=15h_1+20h_2+12h_3;$$
$$D_4=h_1+4h_2+4h_3;\ D_5=3h_1+4h_2+4h_3;\ D_6=h_4+2h_5; \tag{3.2}$$
$$D_7=5h_4+6h_5;\ D_8=h_4+2h_5;\ D_9=D_{10}=h_6.$$

We find the functions χ_n by substituting expressions (2.5) and (2.7) into the first of the inequalities (3.2):

$$\langle\rho\rangle c^2\chi_1=\frac{5}{16x^4}[42u+(23-16k^2-7k^4)x^2]+\frac{15v_2}{8x^7}[7+14vx^2+$$
$$+(7+10k^2+7k^4)x^4]-\frac{15v_4}{8x^7}(7+28x^2+24x^4)-v_{30}^2(1+11x^2+20x^4)+$$
$$+v_{15}-v_3v_{16}+v_5v_{17},$$
$$\langle\rho\rangle c^2\chi_2=\frac{1}{16x^4}[-30u+(-13+8k^2+5k^4)x^2]-\frac{3v_2}{8x^7}[5+10vx^2+$$
$$+(5+6k^2+5k^4)x^4]+\frac{3v_4}{8x^7}(5+20x^2+16x^4)+v_6v_{16}-v_7v_{17},\ \langle\rho\rangle c^2\chi_3=$$
$$=\frac{u}{16x^4}(6+vx^2)+\frac{v_2}{8x^7}[3+6vx^2+(3+2k^2+3k^4)x^4]-\frac{v_4}{8x^7}(3+12x^2+8x^4)-$$
$$-v_8v_{16}+v_9v_{17},\ \langle\rho\rangle c^2\chi_4=\frac{3k^2}{2x^3}(x-2v_2)-v_{15}+v_{10}v_{16},$$
$$\langle\rho\rangle c^2\chi_5=\frac{k^2}{2x^3}(2v_2-x)-v_{11}v_{16},\ \frac{\langle\rho\rangle}{2}\chi_6=-\frac{15u}{8x^2}+v_{30}^2(1+5x^2+12x^4)+ \tag{3.3}$$
$$+\frac{5}{x^2k^3}(k^2v_4v_{25}-v_2v_{10})-v_{32}+v_{16}v_{19}-v_{17}v_{20},\ \frac{\langle\rho\rangle}{2}\chi_7=\frac{3u}{8x^2}+\frac{1}{k^2x^2}(v_2v_{10}-$$
$$-k^2v_4v_{25})-v_{16}v_{21}+v_{17}v_{22},\ \frac{\langle\rho\rangle}{2}\chi_8=v_{32}-\frac{k^2v_{16}}{x},\ \frac{\langle\rho\rangle}{c^2}\chi_9=\frac{3}{x^3}(v_2-v_4)-$$
$$-v_{30}^2x^2(3+4x^2)+v_{33}-v_{16}v_{24}+v_{17}v_{25},$$
$$\frac{\langle\rho\rangle}{c^2}\chi_{10}=\frac{1}{x^3}(v_4-v_2)-v_{33}+v_{16}v_{26}-v_{17}v_{27},$$

where

$$v_{15}=w^2k^2[1+(4+7k^2)x^2+(6+11k^2+3k^4)x^4+$$
$$+(4+k^2-2k^4-3k^6)x^6+(1-3k^2+3k^4-k^6)x^8];$$
$$v_{16}=wx[1-(k^2-1)x^2];\ v_{17}=xv_{30};\ v_{32}=w^2k^2[1+$$
$$+(4+k^2)x^2+(6+7k^2-k^4)x^4+(2+3k^2-6k^4+k^6)x^6+ \tag{3.4}$$
$$+(1-3k^2+3k^4-k^6)x^8];\ v_{33}=w^2k^2x^2[3+(7-3k^2)x^2+$$
$$+(5+2k^2-7k^4)x^4+(1-3k^2+3k^4-k^6)x^6].$$

The remaining values of v_i are determined from expressions (2.7).

Thus, the dimensionless scattering coefficient of longitudinal waves $a\gamma_l(x_l)$ is determined by formulas (1.9), by the third of formulas (1.10), by the first of formulas (3.1), and by formulas (3.2), (1.5), (1.6), (2.6), (2.7), and (2.3). The rate of propagation of longitudinal waves $c_l(x_l)$ is determined by formulas (1.10), by the third of formulas (1.9), by the second of formulas (3.1), and by formulas (2.3), (3.2), (1.5), (1.6), (3.3), (2.7), and (3.4).

§4. If in formulas (2.6), (2.7), (3.3), and (3.4) we pass to the limit for $x \ll 1$ (long waves) and for $x \gg 1$ (short waves), we obtain the well-known limit formulas [6]

$$\gamma_l=\frac{a^3\omega^4}{15\langle\lambda+2\mu\rangle^2}\left[\frac{15h_1+20h_2+12h_3}{c_l^4}+5h_6+\frac{2c_l^3}{c_t^3}\left(\frac{4h_3}{c_t^2c_l^2}+5h_6\right)\right]; \tag{4.1}$$

$$c_l(a\omega)=c_l(1-b_1-4\pi a^2\omega^2b_2)\ (x\ll 1),$$
$$b_1=\frac{1}{30\langle\rho\rangle^2c_l^2}\left[\frac{8h_3}{c_t^2}+\frac{1}{c_l^2}(15h_1+20h_2+12h_3)\right], \tag{4.2}$$
$$b_2=\frac{1}{8\pi\langle\rho\rangle^2}\left\{h_6\left(\frac{2}{c_t^2}+\frac{1}{c_l^2}\right)+\frac{1}{35c_l^2}\left[\frac{56h_3}{c_t^4}+\frac{8h_3}{c_t^2c_l^2}+\frac{1}{c_l^4}(105h_1+196h_2+\right.\right.$$

$$+ 132h_3)\Big] - \frac{2}{5c_l^2}\left[\frac{4h_5}{c_t^2} + \frac{1}{c_l^2}(6h_5 + 5h_4)\right]\Big\};$$

$$\gamma_l = \frac{a\omega^2}{4\langle\lambda + 2\mu\rangle^2}\left(\frac{h_1 + 4h_2 + 4h_3}{c_l^2} + c_l^2 h_6 - 2h_4 - 4h_5\right) \quad (x \gg 1). \tag{4.3}$$

In the short-wave region, there is no dispersion in the given approximation. These limit formulas and the overall formulas were used to calculate the dimensionless scattering coefficient $a\gamma_l(x_l)$ and the rate of propagation of longitudinal ultrasonic waves $c_l(x_l)$ in a tungsten–copper composite material. The densities of tungsten and copper, respectively, were taken [12] equal to $\rho_1 = 19.3$ g/cm^3 and $\rho_2 = 8.9$ g/cm^3, and the volumetric concentrations $v_1 = 0.7$ and $v_2 = 0.3$ [13]. The elastic constants for single crystals of tungsten and copper (in the units 10^{11} dyn cm^2) are equal [14] to $c_{11}^{(1)} = 50.1$; $c_{12}^{(1)} = 19.8$; $c_{44}^{(1)} = 15.14$; $c_{11}^{(2)} = 16.84$; $c_{12}^{(2)} = 12.14$; $c_{44}^{(2)} = 7.54$; they are used to find the mean values of the Lamé constants by means of the relationships [8]

$$\lambda_1 = \frac{1}{5}(c_{11}^{(1)} + 4c_{12}^{(1)} - 2c_{44}^{(1)}); \; \mu_1 = \frac{1}{5}(c_{11}^{(1)} - c_{12}^{(1)} + 3c_{44}^{(1)})$$

and analogously for λ_2 and μ_2.

The results of the calculations are given in Figs. 1 and 2. In Fig. 1, in the region $x_l \geq 0.1$, curves 1 and 2, calculated, respectively, using the overall formulas of the present work and the limit formula for long waves (4.1), differ considerably. Curve 3, plotted using the limit formula for short waves (4.3), in the region $x_l \geq 50$ completely coincides with curve 1. Curve 1 for $x_l \geq 0.5$ is given by dashes, since, for large values of x_l, the Born approximation, used in the calculation of the overall and limit formulas, is found to be inapplicable. For an evaluation of the applicability of the method, we take into consideration that the intensity of a scattered wave must be far less than that of the incident wave: $I_r \ll I_0$. Taking into consideration that, for $r = L$ (L is the length of the sample), $I_r = I_0[1 - \exp(-2\gamma L]$, we find that the condition for the applicability of the Born approximation will be $2\gamma L \ll 1$. If it is taken into account that, in the length of the sample, there must be a sufficient number of grains of inhomogeneity, and if it is assumed that $L \geq 10a$, as well as that $2\gamma L = 0.1$, then, for the scattering coefficient, we obtain the condition for the application of the theory $a\gamma \leq 0.005$, which, for longitudinal waves in the composite material under consideration, corresponds to a value of $x_l \leq 0.5$. From this, for a tungsten–copper composite material, the region of applicability of the calculation made is determined by the condition $\lambda_l \gtrsim 2\pi a/0.5 \sim 10a$.

For the rate of propagation of longitudinal waves (Fig. 2) in the region $x_l \geq 0.1$, the long-wave asymptote 2 [plotted using the limit formula for long waves (4.2)] and curve 1 (plotted using the overall formula) also differ considerably. In the short-wave region with $x_l \geq 10$, the velocity no longer depends on the frequency. It can be seen that, in the limiting case $qa \to 0$, the rate is determined by the effective parameters of the medium; then, as with $qa \to \infty$, the velocity of the longitudinal waves must be equal to the mean velocity. It can be shown by a direct calculation that $c_l(x_l)$ as $qa \to \infty$ is less for a medium with the mean parameters $\langle\lambda + 2\mu\rangle$ and $\langle\rho\rangle$, i.e., for short waves, curve 4 must always lie below c_l (curve 3).

In the region of applicability of the calculation made, the dispersion of the velocity in a tungsten–copper composite material is ~2%.

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VIBRATIONS OF AN ELASTIC INHOMOGENEOUS SOLID WEAKENED BY A CIRCULAR SLIT

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UDC 534.539.3

The vibrations of an elastic homogeneous solid, weakened by a circular slit, were discussed in [1]. A solution of the corresponding static problem was set forth in [2, 3]. For a medium whose Lamé parameters and density depend on the coordinate z, the analogous problem is complicated considerably and admits of an effective analytical solution only for certain cases of the dependence of the above functions on z and of fixed values of the Poisson coefficient.

The present article discusses the static and dynamic problems of determining the displacement in an inhomogeneous elastic solid weakened by a circular slit.

§ 1. We consider a solid elastic medium, occupying the whole space. The Lamé parameters λ and μ and the density of the medium ρ depend on z:

$$\mu = \mu_0(a|z| + 1)^{3-4\nu}, \quad \rho = \rho_0(a|z| + 1)^{4(1-2\nu)}, \tag{1.1}$$

where ν is the Poisson coefficient, assumed constant. As is shown in [4], the equations of motion of such a medium in the case of axial symmetry, in a cylindrical system of coordinates, can be written in the form

$$\begin{gathered} \nabla^2\Phi - v_1^{-2}\varepsilon^{1-4\nu}\frac{\partial^2\Phi}{\partial t^2} = 0, \\ \nabla^2\psi - v_2^{-2}\varepsilon^{1-4\nu}\frac{\partial^2\psi}{\partial t^2} = 0, \end{gathered} \tag{1.2}$$

where $\nabla^2 = \partial^2/\partial r^2 + \partial/r\partial r + \partial^2/\partial z^2$; v_1 and v_2 are the velocities of the deformation waves for $z = 0$, $\varepsilon = a|z| + 1$. The functions Φ and ψ are connected with the vector displacement $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 = u_r\mathbf{i}_r + u_z\mathbf{i}_z$ by the dependences

$$\mathbf{u}_1\varepsilon^{2(1-2\nu)} = \nabla\Phi, \quad \mathbf{u}_2\varepsilon^{2(1-2\nu)} = \nabla \times (\mathbf{i}_\varphi\partial\psi/\partial r), \tag{1.3}$$

where $\mathbf{i}_\varphi$ is a unit vector.

In the plane $z = 0$ there is a circular opening of radius $r = 1$ with its center at the origin of coordinates. It is required to solve the system of equations (1.2) under the assumption that the displacements and stresses in the vicinity of the slit are the same as in a semiinfinite body $z \geq 0$, where, at the free surface $z = 0$, the following boundary conditions obtain:

$$\begin{gathered} \sigma_z = -p_s - p_0\exp(-i\omega t), \ 0 \leqslant r < 1, \\ \tau_{rz} = 0, \ 0 \leqslant r < \infty, \ u_z = 0, \ r > 1, \end{gathered} \tag{1.4}$$

Sumy. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 152-161, May-June, 1978. Original article submitted April 7, 1977.

 0021-8944/78/1903-0404$07.50